

Comments on multiple M2-branes

Jonathan Bagger

*Department of Physics and Astronomy, Johns Hopkins University,
3400 North Charles Street, Baltimore, MD 21218, U.S.A.*

E-mail: bagger@jhu.edu

Neil Lambert

*Department of Mathematics, King's College London,
The Strand, London WC2R 2LS, U.K.*

E-mail: neil.lambert@kcl.ac.uk

ABSTRACT: Recently a three-dimensional field theory was derived that is consistent with all the symmetries expected of the worldvolume action for multiple M2-branes. In this note we examine several physical predictions of this model and show that they are in agreement with expected M2-brane dynamics. In particular, we discuss the quantization of the Chern-Simons coefficient, the vacuum moduli space, a massive deformation leading to fuzzy three-sphere vacua, and a possible large n limit. In this large n limit, the fuzzy funnel solution correctly reproduces the mass of an M5-brane.

KEYWORDS: Brane Dynamics in Gauge Theories, M-Theory, Extended Supersymmetry, Supersymmetric Effective Theories.

Contents

1. Introduction	1
2. The field theory	2
3. Quantizing f^{abcd}	3
4. Vacuum moduli space	4
5. BPS states and a mass deformation	6
6. The large n limit	9
7. Conclusions	11
A. Equivalence with ref. [4]	11

1. Introduction

M-branes are mysterious objects (see e.g. [1]) and virtually nothing is known about their underlying dynamics beyond the case of a single brane. This is in sharp contrast to D-branes [2], where a microscopic description in terms of open strings has driven a huge amount of progress in string theory and gauge theory.

In [3] a model for multiple M2-branes was proposed in which the scalar fields take values in an algebra that admits a totally antisymmetric tri-linear product. It was conjectured that this model could be made maximally supersymmetric by including a non-propagating gauge field. The corresponding supersymmetry algebra was shown to close in [4](v4) and then in [5], where the full equations of motion and Lagrangian were given. The theory is consistent with all the symmetries expected from multiple M2-branes.

Given a theory with the symmetries of a multiple M2-branes, it is natural to see if it reproduces other properties expected of such systems. In this paper we will consider various consequences of the Lagrangian presented in [5]. We will find that several predictions are consistent with expectations, although some aspects of the algebra are not sufficiently well developed to check them all.

The rest of this paper is organized as follows. In section 2 we review the results of ref. [5]. In section 3 we show that consistency requires quantization of the structure constants associated with the tri-linear product. This suggests that the theory is conformally invariant to all orders in perturbation theory. In section 4 we examine the vacuum moduli space of the simplest nontrivial model and argue that, surprisingly, it describes three

M2-branes. In section 5 we consider a mass deformation of the M2-brane worldvolume and show that it leads to fuzzy sphere vacua, as argued in [6]. In section 6 we propose an algebra for an infinite number of M2-branes and show that, when combined with the quantization conditions in section 3, it reproduces the correct energy for the supersymmetric fuzzy funnel solutions of [7]. In the appendix we demonstrate that the approaches of [4] and [5] are equivalent, despite their apparently different algebraic structures.

2. The field theory

The field theory derived in [5] assumes that the scalars X^I , $I = 3, 4, \dots, 10$, and fermions Ψ , $\Gamma_{012}\Psi = -\Psi$, take values in a so-called three-algebra \mathcal{A} . This is a vector space with basis T^a , $a = 1, \dots, N$, that is endowed with a trilinear antisymmetric product

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d, \tag{2.1}$$

from which it is clear that $f^{abc}{}_d = f^{[abc]}{}_d$. We further suppose there is trace-form that provides a metric

$$h^{ab} = \text{Tr} (T^a, T^b), \tag{2.2}$$

which we assume to be positive definite. This allows us to raise and lower indices: $f^{abcd} = f^{abc}{}_e h^{ed}$.

We require two conditions on the triple product. The first is the fundamental identity

$$[T^a, T^b, [T^c, T^d, T^e]] = [[T^a, T^b, T^c], T^d, T^e] + [T^c, [T^a, T^b, T^d], T^e] + [T^c, T^d, [T^a, T^b, T^e]], \tag{2.3}$$

for all $a, b = 1, \dots, N$. This is equivalent to

$$f^{efg}{}_d f^{abc}{}_g = f^{efa}{}_g f^{bcg}{}_d + f^{efb}{}_g f^{cag}{}_d + f^{efc}{}_g f^{abg}{}_d. \tag{2.4}$$

The second is

$$\text{Tr} (T^a, [T^b, T^c, T^d]) = -\text{Tr} ([T^a, T^b, T^c], T^d), \tag{2.5}$$

for all $a, b = 1, \dots, N$. This implies that the f^{abcd} are totally antisymmetric,

$$f^{abcd} = f^{[abcd]}. \tag{2.6}$$

We augment this algebra by including an element T^0 that has a vanishing triple product with everything, i.e. that satisfies $f^{0ab}{}_d = 0$. Assuming $h^{0b} = 0$ when $b \neq 0$, we find $f^{abc}{}_0 = 0$. Thus this mode decouples and can be interpreted as the centre-of-mass coordinate.

There is a natural gauge symmetry on the fields X^I_d , where $\delta X^I_d = \Lambda_{ab} f^{abc}{}_d X^I_c \equiv \tilde{\Lambda}^c{}_d X^I_c$. It gives rise to a covariant derivative $D_\mu X^I_d = \partial_\mu X^I_d - \tilde{A}_\mu{}^c{}_d X^I_c$, with $\delta \tilde{A}_\mu{}^c{}_d = D_\mu \tilde{\Lambda}^c{}_d$, as well as a gauge-covariant field strength $\tilde{F}_{\mu\nu}{}^c{}_d$. The space of all $\tilde{\Lambda}^c{}_d$ is closed under the ordinary matrix commutator, so it generates a matrix Lie algebra \mathcal{G} . From this perspective, $\tilde{A}_\mu{}^c{}_d$ is the usual gauge connection in the adjoint representation of \mathcal{G} , while

the elements of \mathcal{A} are in the fundamental representation. The fundamental identity implies that f^{abcd} is an invariant 4-form of \mathcal{G} .

The Lagrangian derived in [5] is

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}D^\mu X^{aI}D_\mu X_a^I + \frac{i}{2}\bar{\Psi}^a\Gamma^\mu D_\mu\Psi_a + \frac{i}{4}\bar{\Psi}_b\Gamma_{IJ}X_c^IX_d^J\Psi_a f^{abcd} \\ & -V(X) + \frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3}f^{cda}{}_g f^{efgb}A_{\mu ab}A_{\nu cd}A_{\lambda ef}\right), \end{aligned} \tag{2.7}$$

where

$$V(X) = \frac{1}{12}\text{Tr}([X^I, X^J, X^K], [X^I, X^J, X^K]) \tag{2.8}$$

and $\tilde{A}_\mu{}^c{}_d = A_{\mu ab}f^{abc}{}_d$. The Lagrangian is invariant, up to boundary terms, under the transformations

$$\begin{aligned} \delta X_a^I &= i\bar{\epsilon}\Gamma^I\Psi_a \\ \delta\Psi_a &= D_\mu X_a^I\Gamma^\mu\Gamma^I\epsilon - \frac{1}{6}X_b^IX_c^JX_d^K f^{bcd}{}_a\Gamma^{IJK}\epsilon \\ \delta\tilde{A}_\mu{}^b{}_a &= i\bar{\epsilon}\Gamma_\mu\Gamma_I X_c^I\Psi_d f^{cdb}{}_a, \end{aligned} \tag{2.9}$$

where $\Gamma_{012}\epsilon = \epsilon$. These transformations close into translations and gauge transformations. Thus the action has 16 supersymmetries. It also has a manifest SO(8) R-symmetry that acts on the scalars X^I . Furthermore, the action preserves parity if f^{abcd} is taken to be parity odd. These are precisely the symmetries that are expected of the worldvolume description of multiple M2-branes.

This action provides a non-Abelian generalization of the single M2-brane action and describes M2-branes propagating in a flat eleven-dimensional spacetime. As such, it presumably arises as the lowest-order term in a derivative expansion in static gauge of some new κ -symmetric action that generalizes the Born-Infeld action of D-branes. It would be interesting to study this in more detail; however to date, non-Abelian κ symmetry is poorly understood. Here we are compelled to test the predictions of this model against various expectations for multi M2-branes.

3. Quantizing f^{abcd}

Classically, given any choice of structure constants that satisfies the conditions of a three-algebra, namely (2.4) and (2.6), one can rescale the f^{abcd} and preserve the defining conditions. In a quantum theory, however, the coefficient of a Chern-Simons term must be quantized. Therefore, for the case at hand, we expect such a constraint on the f^{abcd} .

To proceed further, we observe that f^{abcd} defines a linear map acting on the vector space of antisymmetric $N \times N$ matrices,

$$f(M_{ab}) = \frac{1}{2}f_{ab}{}^{cd}M_{cd}, \tag{3.1}$$

where we use h^{ab} and its inverse to raise and lower indices. Using the natural inner product, $\langle M^1, M^2 \rangle = M_{ab}^1 M^{2ab}$, one sees that the map is real and symmetric. Therefore it can be diagonalized with eigenvalues that we denote by λ . Using the fundamental identity, one can show that if M_{ab}^1 and M_{ab}^2 are matrices with eigenvalues λ_1 and λ_2 , respectively, then

$$f\left([\tilde{M}^1, \tilde{M}^2]\right) = \lambda_2 [\tilde{M}^1, \tilde{M}^2], \tag{3.2}$$

where $\tilde{M}^c{}_d = f^{abc} M_{ab}$ and $[\tilde{M}^1, \tilde{M}^2]$ is the ordinary matrix commutator. Thus each eigenspace of f is closed under commutation and defines a Lie subalgebra $\mathcal{G}_\lambda \subset \mathcal{G}$.

It follows from (3.2) that

$$f\left([\tilde{M}^1, \tilde{M}^2]\right) = -f\left([\tilde{M}^2, \tilde{M}^1]\right) = -\lambda_1 [\tilde{M}^2, \tilde{M}^1] = \lambda_1 [\tilde{M}^1, \tilde{M}^2], \tag{3.3}$$

and hence

$$\lambda_1 [\tilde{M}^1, \tilde{M}^2] = \lambda_2 [\tilde{M}^1, \tilde{M}^2]. \tag{3.4}$$

This shows that $[\tilde{M}^1, \tilde{M}^2] = 0$ if $\lambda_1 \neq \lambda_2$. Thus the various sub-algebras \mathcal{G}_λ commute and \mathcal{G} decomposes as

$$\mathcal{G} = \oplus_\lambda \mathcal{G}_\lambda. \tag{3.5}$$

As a result of this fact, we can choose a basis in which the twisted Chern-Simons term is

$$\sum_{\lambda \neq 0} \frac{1}{4\lambda} \text{Tr} \left(\tilde{A}^{(\lambda)} \wedge d\tilde{A}^{(\lambda)} + \frac{2}{3} \tilde{A}^{(\lambda)} \wedge \tilde{A}^{(\lambda)} \wedge \tilde{A}^{(\lambda)} \right), \tag{3.6}$$

where $\tilde{A}^{(\lambda)} = \tilde{A}_\mu^{(\lambda)} dx^\mu$ is the projection of the gauge field onto the eigenspace \mathcal{G}_λ , and ordinary matrix multiplication is understood to apply. It is well known that for the path integral to be well-defined, the coefficient of a Chern-Simons term must be $k/4\pi$, where $k \in \mathbb{Z}$ [8] is called the level. Thus we see that the eigenvalues of f must satisfy

$$\lambda = \frac{\pi}{k} \tag{3.7}$$

for each λ , with $k \in \mathbb{Z}$. In the quantum theory, there is no freedom to rescale the f^{abcd} . For simplicity, in the rest of this paper we only consider the case $k = 1$. It would be interesting to examine the physical interpretations of other values of k .

Note that the quantization of f^{abcd} suggests that there are no continuous parameters in the theory. If so, the theory must be conformally invariant to all orders in perturbation theory; since there are no coupling constants, there are no parameters to run. Supersymmetry determines them once and for all.

4. Vacuum moduli space

To explore the connection between our theory and multiple M2-branes, it is natural to start with the vacuum moduli space. Setting $\tilde{A}_\mu = \Psi = \partial_\mu X^I = 0$, the requirement that all supersymmetries be preserved implies that

$$[X^I, X^J, X^K] = 0, \tag{4.1}$$

for all X^I . This condition also ensures that the equations of motion are satisfied.

Let us focus on the simplest nontrivial possibility, in which the three-algebra \mathcal{A} has four generators and hence, given the quantization condition found above,

$$f^{abcd} = \pi \varepsilon^{abcd}, \tag{4.2}$$

where $a, b = 1, 2, 3, 4$. Without loss of generality, we take $h^{ab} = \delta^{ab}$. We call this three-algebra \mathcal{A}_4 ; one can check that it satisfies the fundamental identity. The solutions to the vacuum equations (4.1) are given by

$$X^I = a^I \alpha + b^I \beta, \tag{4.3}$$

where α and β are any two elements of \mathcal{A}_4 .

We next consider the gauge transformations. For the case at hand, the Lie algebra \mathcal{G}_4 is generated by $\varepsilon^{abc}{}_d \Lambda_{ab}$, where Λ_{ab} is real and antisymmetric. Thus \mathcal{G}_4 is nothing but the set of all antisymmetric real 4×4 matrices, i.e. $\mathcal{G}_4 = so(4) \equiv so(3) \oplus so(3)$. The elements $\alpha, \beta \in \mathcal{A}_4$ are in the fundamental representation of $SO(4)$. Therefore, up to a gauge transformation, we can set $\alpha \propto T^1$. Furthermore, using the little group $SO(3)$ of T^1 we can also choose $\beta \propto T^2$. Thus, up to a gauge transformation, the vacuum moduli space is parameterized by

$$X^I = a^I T^1 + b^I T^2. \tag{4.4}$$

This result implies that there are two bosonic zero modes for each of the coordinates X^I . Including the overall center-of-mass generator T^0 , which decouples from all the interactions and gauge symmetries, we find three bosonic zero modes for each scalar X^I . As with multiple D-branes, M2-branes satisfy a no-force condition and hence the most natural interpretation for these zero modes is that they correspond to moving the M2-branes apart in transverse directions. Therefore we are led to identify the Lagrangian with the worldvolume theory of three M2-branes. Note that this argument assumes that our construction describes a generic point in the moduli space; at special points there may be fewer degrees of freedom.¹

It seems peculiar that the simplest nontrivial model describes three M2-branes, rather than two. Let us therefore make some comments as to why this might be the case. If we think of the worldvolume theory of n M2-branes as the IR fixed point of three-dimensional $U(n)$ super-Yang-Mills theory, then we expect fewer than n^2 degrees of freedom per field in the IR. The smallest three-algebra must have at least four generators, and including the center-of-mass gives a total of five degrees of freedom per field. Thus one would not expect this algebra to arise as the IR fixed point of two D2-branes, but rather as the fixed point of three D2-branes. We note that the number N of degrees of freedom of n M2-branes is conjectured to scale as $N = n^{3/2}$ at large n , and it is encouraging to observe that $3^{3/2} \sim 5.2$.

A possible resolution is that the IR fixed point of two D2-branes is trivial. It is natural to expect that the worldvolume theory of two D2-branes loses at least one degree of freedom in the IR, leaving at most three. Factoring out the center-of-mass would then leave at most

¹We are grateful to M. van Raamsdonk for bringing this point to our attention.

two interacting degrees of freedom. This might be too few to construct a theory that is consistent with all the symmetries (even without assuming a Lagrangian description). In particular, it is too small to identify the fields with elements of a Lie algebra.

A more detailed analysis of the degrees of freedom requires finding a class of three-algebras with arbitrarily large dimension N . At present we do not know of any other finite-dimensional cases. However, we can make some observations. For $N > 4$ the map f must have a nontrivial kernel, for the following reason. Suppose it has a trivial kernel. Then the space \mathcal{G} of all $\tilde{\Lambda}^c_d = f^{abc}_d \Lambda_{ab}$ would be all of $so(N)$. In that case f^{abcd} would be an invariant four-tensor of $so(N)$, but there are no such invariants for $N > 4$. A non-vanishing kernel would lead to additional vacuum moduli and hence a larger number of M2-branes.

5. BPS states and a mass deformation

In ref. [6], it was argued that in the presence of a particular background four-form flux, M2-branes preserve four supersymmetries and exhibit an $SO(4)$ R-symmetry. Furthermore, the flux induces a supersymmetric mass term for the worldvolume scalars and fermions. It was also argued that in this background, the vacuum of n M2-branes is a state in which the scalars describe a fuzzy three-sphere in spacetime. The M2-branes ‘puff up’ so that their worldvolume is of the form $\mathbb{R}^{1,2} \times \tilde{S}^3$, where \tilde{S}^3 is a fuzzy three-sphere that becomes a normal S^3 as $n \rightarrow \infty$. This setup provides an M-theory analog of the Myers effect that occurs for D-branes in the presence of background fluxes [9].

In this section we search for such solutions to our theory. Since we are not interested in the gauge fields and fermions, we truncate the Lagrangian to include only the scalar fields,

$$\mathcal{L}_B = -\frac{1}{2}\text{Tr} (\partial_\mu X^I, \partial^\mu X^I) - \frac{1}{12}\text{Tr} ([X^I, X^J, X^K], [X^I, X^J, X^K]). \quad (5.1)$$

Consistency requires that $X^A_a \partial_\mu X^A_b f^{abc}_d = 0$, which follows from the gauge field equation of motion. This relation is satisfied in all the solutions discussed below.

We search for solutions with four non-vanishing scalars, which we denote by X^A , $A = 1, 2, 3, 4$. The search is simplified by writing the potential in the following form,

$$V(X) = \frac{1}{2}\text{Tr} (\partial^A W, \partial^A W), \quad (5.2)$$

where

$$W = \frac{1}{24}\varepsilon^{ABCD}\text{Tr} (X^A, [X^B, X^C, X^D]) \quad (5.3)$$

is the ‘superpotential.’ We add an $SO(4)$ symmetric mass term by generalizing (5.3) to

$$W = \frac{1}{2}m\text{Tr} (X^A, X^A) + \frac{1}{24}\varepsilon^{ABCD}\text{Tr} (X^A, [X^B, X^C, X^D]). \quad (5.4)$$

Vacuum solutions require $\partial^A W = 0$, or

$$mX^A = -\frac{1}{6}\varepsilon^{ABCD} [X^B, X^C, X^D]. \quad (5.5)$$

In addition to the trivial solution $X^A = 0$, eq. (5.5) has a fuzzy S^3 solution in which the M2's puff up into a fuzzy three-sphere. The two solutions describe two zero-energy vacuum states of the M2-brane in the four-flux background.

To construct the fuzzy three-sphere vacuum, we suppose that the three-algebra admits a representation of \mathcal{A}_4 , so the four generators T^A satisfy $[T^A, T^B, T^C] = \pi \varepsilon^{ABCD} T^D$. The solution is found by taking

$$X^A = \sqrt{\frac{m}{\pi}} T^A, \tag{5.6}$$

with $m > 0$. It describes a fuzzy three-sphere with radius proportional to \sqrt{m} , in agreement with [6]. In the case of D-branes, physically distinct vacua arise from different representations of the symmetry algebra [9]. Presumably, there is a similar family of solutions here, corresponding to different numbers of M2-branes. We will not attempt to discuss them further because we lack a sufficient understanding of three-algebra representations.

We can also construct the BPS fuzzy funnel solutions of [3, 7], in which the M2-branes end on an M5-brane. Following Bogomoln'yi, we consider static solutions that depend on one coordinate $x^2 = s$. We write the energy as

$$\begin{aligned} E &= \frac{1}{2} \int ds dx^1 \text{Tr} \left(\frac{dX^A}{ds} - \partial^A W, \frac{dX^A}{ds} - \partial^A W \right) + 2\partial^A W \frac{dX^A}{ds} \\ &= \frac{1}{2} \int ds dx^1 \text{Tr} \left(\frac{dX^A}{ds} - \partial^A W, \frac{dX^A}{ds} - \partial^A W \right) + 2 \frac{dW}{ds}. \end{aligned}$$

Therefore, up to a boundary term, the minimum energy solutions satisfy

$$\partial_2 X^A = \partial^A W = m X^A + \frac{1}{6} \varepsilon^{ABCD} [X^B, X^C, X^D]. \tag{5.7}$$

The fuzzy funnel solution is found by taking

$$X^A = f(s) T^A, \tag{5.8}$$

where $s = x^2$ and again the T^A satisfy $[T^A, T^B, T^C] = \pi \varepsilon^{ABCD} T^D$. The equation for f is

$$f' = m f - \pi f^3; \tag{5.9}$$

the solution is

$$f = \sqrt{\frac{m}{\pi}} \frac{1}{\sqrt{1 - c e^{-2ms}}}, \tag{5.10}$$

where c is a constant, which by translation can be set to ± 1 .

If $c = +1$ and $m > 0$, the solution behaves as $f = 1/\sqrt{2\pi s}$ for small but positive s . It approaches $f \rightarrow \sqrt{m/\pi}$ as $s \rightarrow \infty$. If $m < 0$, the function f has the same behavior at small and positive s , but $f \rightarrow 0$ as $s \rightarrow \infty$. These solutions describe fuzzy funnels in which an infinite radius fuzzy three-sphere at $s = 0$ relaxes into the fuzzy sphere or the trivial vacuum, respectively, as $s \rightarrow \infty$. The spacetime interpretation of these solutions is that they correspond to M2-branes that end on a single M5-brane, located at $s = 0$ and infinitely extended along the $(x^0, x^1, x^2, x^3, x^4, x^5)$ directions.

On the other hand, if $c = -1$ and $m > 0$, the function f is bounded. It vanishes exponentially as $s \rightarrow -\infty$ and approaches $f \rightarrow \sqrt{m/\pi}$ as $s \rightarrow \infty$. Here there is no divergent fuzzy funnel, i.e. no M5-brane. This solution smoothly interpolates between the trivial and fuzzy sphere vacua. In other words, it is a traditional domain wall that interpolates between two degenerate vacuum solutions of the worldvolume effective action.

We conclude this section by explicitly checking the predictions for the energy of the fuzzy funnel and the physical radius of the fuzzy three-sphere vacuum. We follow [7] and calculate the total (divergent) energy of a fuzzy funnel solution, with $m = 0$,

$$\begin{aligned}
 E &= \int_{-\infty}^{\infty} dx^1 \int_0^{\infty} ds \text{Tr} (\partial_2 X^A, \partial_2 X^A) \\
 &= \text{Tr} (T^A, T^A) \int_{-\infty}^{\infty} dx^1 \int_0^{\infty} ds f'^2 \\
 &= \pi \text{Tr} (T^A, T^A) \int_{-\infty}^{\infty} dx^1 \int_0^{\infty} f^3 df.
 \end{aligned}
 \tag{5.11}$$

Next we introduce the physical fuzzy sphere radius, R , which is defined to be the root mean square radius, averaged over the n M2-branes,

$$\begin{aligned}
 R^2 &= \frac{\text{Tr} (X^A, X^A)}{nT_2} \\
 &= \frac{\text{Tr} (T^A, T^A)}{nT_2} f^2.
 \end{aligned}
 \tag{5.12}$$

Note that we have inserted a factor of the membrane tension, T_2 . This follows from the fact that X^A is canonically normalized and hence has mass dimension 1/2. Thus it cannot be directly interpreted as a spacetime coordinate. Instead, the spacetime coordinates should be identified with $X^A/\sqrt{T_2}$, which has the dimension of length. This change of variable rescales the kinetic term of the action to

$$\mathcal{L} = -\frac{T_2}{2} \text{Tr} (\partial_\mu X^I, \partial^\mu X^I) + \dots,
 \tag{5.13}$$

as expected for a membrane with tension T_2 .

From these expressions we have what we need to compute the energy:

$$\begin{aligned}
 E &= \frac{T_2^2}{2\pi} \frac{n^2}{\text{Tr}(T^A, T^A)} \int_{-\infty}^{\infty} dx^1 \int_0^{\infty} 2\pi^2 R^3 dR \\
 &= T_5 \frac{n^2}{\text{Tr}(T^A, T^A)} \int d^5x,
 \end{aligned}
 \tag{5.14}$$

where we have used the fact that $T_2^2 = 2\pi T_5$ [10]. This expression, at least in the large n limit, should reproduce the energy of an infinite M5-brane with tension T_5 . This implies that

$$\text{Tr} (T^A, T^A) = n^2
 \tag{5.15}$$

at large n . Unfortunately, we do not know enough about the representations of three-algebras to confirm this prediction.

Finally, we return our attention to the fuzzy three-sphere vacuum described above. Using (5.13), we see that in the large n limit, the physical radius is

$$\begin{aligned}
 R^2 &= \frac{\text{Tr}(X^A, X^A)}{nT_2} \\
 &= \frac{m \text{Tr}(T^A, T^A)}{\pi nT_2} \\
 &= \frac{mn}{\pi T_2},
 \end{aligned}
 \tag{5.16}$$

where we have used (5.15). In the units of [6], the tension $T_2 = M_{11}^3/4\pi^2$, and hence

$$R^2 = \frac{4\pi mn}{M_{11}^3}.
 \tag{5.17}$$

This agrees with the result in [6], up to a factor 4/3.

The energy density for the smooth domain wall that arises when $c = -1$ and $m > 0$ can also be calculated. We find

$$\begin{aligned}
 \mathcal{E} &= \int_{-\infty}^{\infty} ds \text{Tr}(\partial_2 X^A, \partial_2 X^A) \\
 &= \text{Tr}(T^A, T^A) \int_{-\infty}^{\infty} ds f'^2 \\
 &= \frac{m^2}{4\pi} \text{Tr}(T^A, T^A) \\
 &= \frac{m^2 n^2}{4\pi},
 \end{aligned}
 \tag{5.18}$$

where the last line assumes the large- n relation (5.15).

6. The large n limit

In this section we propose a large n limit for the three-algebra \mathcal{A} . A natural infinite-dimensional example of a three-algebra is given by the space $\mathcal{C}^\infty(\Sigma)$ of differentiable functions on a closed three-manifold Σ endowed with a metric. For simplicity we assume that Σ is compact without boundary and with a finite volume. In this case the triple product is given by the Nambu bracket [11]

$$[X, Y, Z] = -\pi \star (dX \wedge dY \wedge dZ).
 \tag{6.1}$$

It can be shown that (6.1) satisfies the fundamental identity. Furthermore, if we take

$$\text{Tr}(X, Y) = \frac{1}{\text{vol}(\Sigma)} \int_{\Sigma} X \wedge \star Y,
 \tag{6.2}$$

then (6.1) also satisfies (2.5). Note that we have normalized the trace-form so that the identity function has unit length (and can be identified with the translational generator T^0).

For this three-algebra, the gauge symmetry generated by the Nambu bracket is

$$\begin{aligned}\delta X &= [\alpha, \beta, X] \\ &= v^k \partial_k X,\end{aligned}\tag{6.3}$$

where $v^k = -(\pi/\sqrt{g})\varepsilon^{ijk}\partial_i\alpha\partial_j\beta$ and σ^i , $i = 1, 2, 3$ are local coordinates on Σ . This transformation is nothing but an area-preserving diffeomorphism on Σ .

We wish to consider the large n limit of the fuzzy three-sphere vacua found in the previous section. To do this we need to find a representation of \mathcal{A}_4 inside $\mathcal{C}^\infty(\Sigma)$. Since we need an $so(4)$ symmetry it is natural to take $\Sigma = S^3$, the unit sphere inside \mathbb{R}^4 . We then consider the four functions T^A that describe the natural embedding of S^3 into \mathbb{R}^4 :

$$\begin{aligned}T^1 &= \cos \theta_1 \\ T^2 &= \sin \theta_1 \cos \theta_2 \\ T^3 &= \sin \theta_1 \sin \theta_2 \cos \theta_3 \\ T^4 &= \sin \theta_1 \sin \theta_2 \sin \theta_3.\end{aligned}\tag{6.4}$$

One finds that these functions satisfy (see also [12])

$$[T^A, T^B, T^C] = \pi \varepsilon^{ABCD} T^D,\tag{6.5}$$

and also $\text{Tr}(T^A, T^B) = \frac{1}{4}\delta^{AB}$. Thus the functions T^A provide a representation of \mathcal{A}_4 inside $\mathcal{C}^\infty(S^3)$.

We now return to the fuzzy funnel solution. In the infinite n limit, we expect that the fuzzy sphere loses its ‘fuzziness.’ We define the physical radius to be (*cf* (5.13))

$$R^2 = \frac{\text{Tr}(X^A, X^A)}{T_2}.\tag{6.6}$$

Following the calculations of the previous section, we find that the energy of the fuzzy funnel is

$$E = T_5 \int d^5x,\tag{6.7}$$

which exactly reproduces the tension of an M5-brane.

It seems natural to propose that this three-algebra is the large n limit of the finite dimensional three-algebras that describe n M2-branes. It is tempting to further speculate that the three-manifold Σ should somehow be identified with the worldvolume of the M2-branes (or possibly with the worldvolume of an open M2-brane that plays a role analogous to the one that open strings play in the definition of D-branes). The gauge symmetries are then simply the area preserving diffeomorphisms of the M2-brane worldvolume. Note that the gauge field is non-dynamical with a Chern-Simons-like kinetic term, and that this is consistent with identifying it with the metric in three dimensions. Is it intriguing to note that area-preserving diffeomorphism have previously been associated with the gauge symmetry of M2-branes [13].

Furthermore, we observe that if Σ is the worldvolume of the M2-branes, then under a parity transformation the triple product (6.1) changes sign. This would then explain why one needs to have $f^{abcd} \rightarrow -f^{abcd}$ in the finite dimensional cases to preserve parity.

7. Conclusions

In this paper we have analyzed various physical aspects of the multiple M2-brane Lagrangian proposed in [5]. In particular, we discussed the quantization of f^{abcd} that is required by the quantum theory, the vacuum moduli space of the simplest example, and various features of fuzzy sphere vacua and fuzzy funnels. We also proposed a natural infinite n three-algebra, and showed that it correctly produces the energy density of a fuzzy-funnel solution, with no arbitrary parameters. In so far as we have been able to check, the theory is consistent with all expectations. It would also be interesting to compare this model with predictions from the BFFS matrix model description of M-theory [14].

We believe that the most pressing open issue is obtaining an infinite class of three-algebras that can represent an arbitrary number of M2-branes. There is a large literature on related algebras that arise from quantization of the Nambu bracket, starting with the work of [11, 15]. However, much of this literature imposes slightly different conditions on the triple-product, such as a Leibnitz property that we do not require or a generalized Jacobi identity that is weaker than the fundamental identity (for example see [16]). With such a class of three-algebras, one would presumably be able to analyze the vacuum moduli spaces and deduce the infamous relation $N = n^{3/2}$. (For an alternative derivation, see [17].)

Finally, we note that in this paper we have restricted our attention to the algebraic structure presented in [3, 5]. However, as shown in the appendix, there is an equivalent definition that was introduced in [4]. The relation between the two is worth exploring in greater detail.

Acknowledgments

We would like to thank A. Gustavsson for email correspondence and M. van Raamsdonk for comments. JB is supported in part by the US National Science Foundation, grant NSF-PHY-0401513. NL is supported in part by the PPARC grant PP/C507145/1 and the EU grant MRTN-CT-2004-512194.

A. Equivalence with ref. [4]

In [4], Gustavsson presented an algebraic structure in which there are two vector spaces \mathcal{A} and \mathcal{B} . For $\alpha, \beta \in \mathcal{A}$ and $A, B \in \mathcal{B}$, he considered bi-linear products of the form

$$\begin{aligned} \langle \alpha, \beta \rangle &= -\langle \beta, \alpha \rangle \in \mathcal{B} \\ (A, \alpha) &\in \mathcal{A} \\ [A, B] &= -[B, A] \in \mathcal{B}, \end{aligned} \tag{A.1}$$

as well as the ‘associative’ condition

$$(\langle \alpha, \beta \rangle, \gamma) = (\langle \beta, \gamma \rangle, \alpha). \tag{A.2}$$

He then imposed the ‘Jacobi’ identities

$$\begin{aligned} \langle (A, \alpha), \beta \rangle - \langle (A, \beta), \alpha \rangle &= [A, \langle \alpha, \beta \rangle] \\ (A, (B, \alpha)) - (B, (A, \alpha)) &= ([A, B], \alpha) \\ [[A, B], C] + [B, [A, C]] &= [A, [B, C]]. \end{aligned} \tag{A.3}$$

Note that the final condition is simply the statement that \mathcal{B} is a Lie algebra. With these structures we can construct a tri-linear product on \mathcal{A} ,

$$[\alpha, \beta, \gamma] \equiv (\langle \alpha, \beta \rangle, \gamma). \tag{A.4}$$

Note that this triple product is manifestly antisymmetric in $\alpha \leftrightarrow \beta$ and the associative condition (A.2) further implies that it is totally antisymmetric in α, β, γ .

We will first show that Gustavsson’s structure satisfies the fundamental identity cited in [5]. Using the second Jacobi identity, we find

$$\begin{aligned} [\alpha, \beta, [\gamma, \delta, \epsilon]] &= (\langle \alpha, \beta \rangle, (\langle \gamma, \delta \rangle, \epsilon)) \\ &= (\langle \gamma, \delta \rangle, (\langle \alpha, \beta \rangle, \epsilon)) + ([\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle], \epsilon) \\ &= [\gamma, \delta, [\alpha, \beta, \epsilon]] + ([\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle], \epsilon), \end{aligned} \tag{A.5}$$

The second term on the right-hand side can be rewritten using the first Jacobi identity as

$$\begin{aligned} [\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle] &= \langle (\langle \alpha, \beta \rangle, \gamma), \delta \rangle - \langle (\langle \alpha, \beta \rangle, \delta), \gamma \rangle \\ &= \langle [\alpha, \beta, \gamma], \delta \rangle - \langle [\alpha, \beta, \delta], \gamma \rangle, \end{aligned} \tag{A.6}$$

and hence

$$([\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle], \epsilon) = [[\alpha, \beta, \gamma], \delta, \epsilon] - [[\alpha, \beta, \delta], \gamma, \epsilon]. \tag{A.7}$$

Thus we see that

$$[\alpha, \beta, [\gamma, \delta, \epsilon]] = [\gamma, \delta, [\alpha, \beta, \epsilon]] + [[\alpha, \beta, \gamma], \delta, \epsilon] + [\gamma, [\alpha, \beta, \delta], \epsilon], \tag{A.8}$$

which is the fundamental identity. This proves that the algebraic structure introduced in [4] satisfies the algebraic condition in [5].

To show equivalence, we also need to prove the other way around. Therefore we start with the algebraic structure used in [5], consisting of a single vector space \mathcal{A} with elements α, β, \dots , and a tri-linear totally antisymmetric product

$$[\alpha, \beta, \gamma] \in \mathcal{A} \tag{A.9}$$

that satisfies the fundamental identity

$$[\alpha, \beta, [\gamma, \delta, \epsilon]] = [[\alpha, \beta, \gamma], \delta, \epsilon] + [\gamma, [\alpha, \beta, \delta], \epsilon] + [\gamma, \delta, [\alpha, \beta, \epsilon]], \tag{A.10}$$

and show that how to construct bilinear products that obey the relations (A.1) – (A.3).

For $X \in \mathcal{A}$, we can define a vector space of linear maps from \mathcal{A} to itself, generated by

$$as_{\alpha,\beta}(X) = [\alpha, \beta, X], \quad (\text{A.11})$$

This is the space \mathcal{B} , with elements generated by $A = as_{\alpha,\beta}$. (In [5], this space was denoted by \mathcal{G} .) One sees, using the fundamental identity, that

$$[\alpha_1, \beta_1, [\alpha_2, \beta_2, X]] - (1 \leftrightarrow 2) = [[\alpha_1, \beta_1, \alpha_2], \beta_2, X] + [\alpha_2, [\alpha_1, \beta_1, \beta_2], X]. \quad (\text{A.12})$$

This shows that $as_{\alpha_1,\beta_1} \circ as_{\alpha_2,\beta_2} - as_{\alpha_2,\beta_2} \circ as_{\alpha_1,\beta_1}$ is again an element of \mathcal{B} . In fact, linear maps of a vector space to itself are associative under composition, so \mathcal{B} is a Lie algebra using the ordinary commutator, satisfying the Jacobi identity (which is the final condition in (A.3)). Equation (A.12) also shows that

$$[A_1, A_2](X) = [[\alpha_1, \beta_1, \alpha_2], \beta_2, X] + [\alpha_2, [\alpha_1, \beta_1, \beta_2], X], \quad (\text{A.13})$$

where A_i denotes the map $A_i(X) = as_{\alpha_i,\beta_i}(X)$. Thus the right hand side is actually anti-symmetric in $A_1 \leftrightarrow A_2$.

The rest of algebraic structure introduced in [4] can be constructed as follows. It is natural to define

$$\langle \alpha, \beta \rangle = as_{\alpha,\beta}, \quad (\text{A.14})$$

and

$$(A, \alpha) = as_A(\alpha). \quad (\text{A.15})$$

The condition (A.2) then follows from the antisymmetry of $[\alpha, \beta, \gamma]$. To prove (A.3), we compute

$$\begin{aligned} \langle (A, \alpha), \beta \rangle(X) - \langle (A, \beta), \alpha \rangle(X) &= [as_A(\alpha), \beta, X] - [as_A(\beta), \alpha, X] \\ &= [as_A(\alpha), \beta, X] + [\alpha, as_A(\beta), X] \\ &= [as_A, as_{\alpha,\beta}](X) \\ &= [as_A, \langle \alpha, \beta \rangle](X) \end{aligned} \quad (\text{A.16})$$

for arbitrary X . This reproduces the first condition in (A.3). We then compute

$$\begin{aligned} (A, (B, \alpha)) - (B, (A, \alpha)) &= as_A(as_B(\alpha)) - as_B(as_A(\alpha)) \\ &= [as_A, as_B](\alpha) \\ &= ([A, B], \alpha), \end{aligned} \quad (\text{A.17})$$

which is the second condition in (A.3). Thus a three-algebra that satisfies the fundamental identity also provides an example of the algebraic structure in [4]. This proves that the two approaches are, in fact, equivalent.

References

- [1] D.S. Berman, *M-theory branes and their interactions*, *Phys. Rept.* **456** (2008) 89 [[arXiv:0710.1707](#)].
- [2] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [[hep-th/9510017](#)].
- [3] J. Bagger and N. Lambert, *Modeling multiple M2's*, *Phys. Rev. D* **75** (2007) 045020 [[hep-th/0611108](#)].
- [4] A. Gustavsson, *Algebraic structures on parallel M2-branes*, [arXiv:0709.1260](#).
- [5] J. Bagger and N. Lambert, *Gauge symmetry and supersymmetry of multiple M2-branes*, [arXiv:0711.0955](#).
- [6] I. Bena, *The M-theory dual of a 3 dimensional theory with reduced supersymmetry*, *Phys. Rev. D* **62** (2000) 126006 [[hep-th/0004142](#)].
- [7] A. Basu and J.A. Harvey, *The M2-M5 brane system and a generalized Nahm's equation*, *Nucl. Phys. B* **713** (2005) 136 [[hep-th/0412310](#)].
- [8] S. Deser, R. Jackiw and S. Templeton, *Three-dimensional massive gauge theories*, *Phys. Rev. Lett.* **48** (1982) 975.
- [9] R.C. Myers, *Dielectric-branes*, *JHEP* **12** (1999) 022 [[hep-th/9910053](#)].
- [10] J.H. Schwarz, *The power of M-theory*, *Phys. Lett. B* **367** (1996) 97 [[hep-th/9510086](#)].
- [11] Y. Nambu, *Generalized Hamiltonian dynamics*, *Phys. Rev. D* **7** (1973) 2405.
- [12] D. Sadri and M.M. Sheikh-Jabbari, *Giant hedge-hogs: spikes on giant gravitons*, *Nucl. Phys. B* **687** (2004) 161 [[hep-th/0312155](#)].
- [13] B. de Wit, J. Hoppe and H. Nicolai, *On the quantum mechanics of supermembranes*, *Nucl. Phys. B* **305** (1988) 545.
- [14] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M theory as a matrix model: a conjecture*, *Phys. Rev. D* **55** (1997) 5112 [[hep-th/9610043](#)].
- [15] L. Takhtajan, *On foundation of the generalized Nambu mechanics (second version)*, *Commun. Math. Phys.* **160** (1994) 295 [[hep-th/9301111](#)].
- [16] M.M. Sheikh-Jabbari, *Tiny graviton matrix theory: dLCQ of IIB plane-wave string theory, a conjecture*, *JHEP* **09** (2004) 017 [[hep-th/0406214](#)].
- [17] D.S. Berman and N.B. Copland, *A note on the M2-M5 brane system and fuzzy spheres*, *Phys. Lett. B* **639** (2006) 553 [[hep-th/0605086](#)].